J. Fluid Mech. (1969), *vol.* 39, *part* 3, *pp.* 465–476
Printed in Great Britain

Low Reynolds number flow of a variable property gas past a heated circular cylinder

By P. F. HODNETT
Department of Aerospace Engineering,
University of Connecticut, Storrs, Connecticut

(Received 6 February 1969 and in revised form 14 July 1969)

The low Reynolds number flow of a variable property gas past an infinite heated circular cylinder is studied when the temperature difference between the cylinder and the free stream is appreciable. The velocity field (and hence the drag on the cylinder) is calculated by the method of matched asymptotic expansions. It is found that the zero-order velocity field calculated on the Stokes approximation satisfies both the no slip condition at the cylinder and the uniform stream condition at infinity which is in strong contrast with the corresponding velocity field for incompressible slow flow past an unheated cylinder where the uniform stream condition at infinity cannot be satisfied. When the temperature of the cylinder is twice the temperature at infinity it is found that the drag on the cylinder is almost twice the drag on a similar unheated cylinder.

1. Introduction

Recently there has been interest in the effects of variable gas properties on low Reynolds number flow phenomena. Thus Chang (1965), Kassoy, Adamson & Messiter (1966) and more recently Rimmer (1968) have, in theoretical analyses, considered the heated sphere in a slow uniform stream. Theoretical investigations of the slow flow of a gas past a heated circular cylinder have also been undertaken. Interest in this latter problem is increased because of the experimental use of the hot-wire anemometer which the theory describes.

The heat transfer from a circular cylinder in a low Reynolds number flow was investigated theoretically by Cole & Roshko (1954) who calculated the zero-order temperature field (and hence the Nusselt number) for a slightly heated cylinder. Later work by Kassoy (1967), who also included slip flow effects in his solution, produced the first- and second-order corrections to the Cole & Roshko solution. More recent work by Wood (1968) and Hieber & Gebhart (1968) essentially confirmed Kassoy's findings, although Hieber & Gebhart included the case of large Prandtl number in their analysis. In all the other work the Prandtl number is taken to be of order one. In the context of the above-mentioned papers, low Reynolds numbers mean Reynolds numbers tending to zero. A numerical analysis of the heat transfer from a slightly heated circular cylinder was undertaken by Dennis, Hudson & Smith (1968) for a range of Reynolds numbers from 0·01 to 40. Comparison of these results with those of Kassoy (1967) showed satisfactory agreement for Reynolds numbers less than 0·5.

The author (Hodnett 1968) calculated the velocity field and hence the drag on the cylinder for this problem of slow flow past a heated circular cylinder when the heating factor is small compared to one (we call the non-dimensionalized temperature difference between the cylinder and free stream the heating factor). In the same paper the velocity field when the heating factor is of order one was obtained when allowing the proper density variation but assuming that the thermal conductivity and viscosity are constant. We here give the correct velocity field (and drag) when the heating factor is order one and when the density, thermal conductivity and viscosity are all allowed their proper variation.

We are therefore considering the flow at low Reynolds numbers ($R_e \to 0$) of a gas past an infinite heated circular cylinder when the axis of the cylinder is perpendicular to the plane of flow. The temperature difference between the cylinder and the free stream is significant. Attention is confined to the continuum régime (i.e. $M_\infty/R_e \sim K_n \to 0$). Here, M_∞ is the Mach number evaluated at infinity and K_n is the Knudsen number. The viscosity and thermal conductivity are taken to vary as $(T')^m$ where T' is the temperature and m is non-negative. The solution is derived by the method of matched asymptotic expansions previously applied by Kaplun (1957) and Proudman & Pearson (1957) to the flow of a low Reynolds number *incompressible* fluid past a circular cylinder. To facilitate the analysis the gas is taken to be perfect with constant specific heats (e.g. any monatomic gas).

2. Equations

Let dashed quantities denote physical variables; these are non-dimensionalized with respect to their free stream values, so that $\rho = \rho'/\rho'_\infty$, $\mu = \mu'/\mu'_\infty$, $K = K'/K'_\infty$, $\mathbf{q} = \mathbf{q}'/\mathbf{U}'_\infty$, $T = T'/T'_\infty$, $r = r'/r'_0$, where ρ' is the density, $\mathbf{q}'$ the velocity, U'_∞ the speed of the free stream, r' the distance from the centre point of the cylinder whose radius is r'_0, and the subscript ∞ denotes the value of a variable at infinity. The pressure p' is non-dimensionalized in two ways as follows:

$$p = \frac{r'_0(p'-p'_\infty)}{\mu'_\infty U'_\infty} \quad \text{and} \quad P = \frac{p'-p'_\infty}{\rho'_\infty U'^2_\infty},$$

so that $p = R_e P$, where $R_e = r'_0 \rho'_\infty U'_\infty/\mu'_\infty$ is the Reynolds number.

The appropriate non-dimensionalized equations for the region near the body, which will be referred to as the Stokes or inner region, are

$$\operatorname{div}(\rho\mathbf{q}) = 0, \tag{1}$$

$$R_e\rho(\tfrac{1}{2}\nabla\mathbf{q}^2 - \mathbf{q}\times\operatorname{curl}\mathbf{q}) = -\nabla p + \tfrac{4}{3}\nabla(\mu\operatorname{div}\mathbf{q})$$
$$+\nabla(\mathbf{q}.\nabla\mu) - \mathbf{q}\nabla^2\mu + \nabla\mu\times\operatorname{curl}\mathbf{q} - (\operatorname{div}\mathbf{q})\nabla\mu - \operatorname{curl}\operatorname{curl}\mu\mathbf{q}, \tag{2}$$

$$R_e\rho(\mathbf{q}.\nabla)T = P_r^{-1}\operatorname{div}(K\nabla T) + (\gamma'-1)M^2_\infty(\mu\phi + \mathbf{q}.\nabla p), \tag{3}$$

while the equation of state is

$$\gamma'\frac{M^2_\infty}{R_e}p + 1 = \rho T, \tag{4}$$

where $P_r = \mu'_\infty C_p/K'_\infty$ is the Prandtl number with C_p the specific heat at constant pressure, $M_\infty = U'_\infty/a'_\infty$ is the value of the Mach number at infinity where a' is the sound speed defined for a perfect gas by $(a')^2 = \gamma' p'/\rho'$ with γ' the ratio of the specific heats; ϕ is the dissipation function given by

$$\phi = \nabla^2(\mathbf{q}.\mathbf{q}) + 2\nabla.(\operatorname{curl}\mathbf{q}\times\mathbf{q}) - 2\mathbf{q}.\nabla(\operatorname{div}\mathbf{q}) + (\operatorname{curl}\mathbf{q}.\operatorname{curl}\mathbf{q}) - \tfrac{2}{3}(\operatorname{div}\mathbf{q})^2.$$

In the operators appearing in (1)–(4) the Stokes variable r is used. Denoting the temperature of the surface of the cylinder by T'_w, we restrict $(T'_w - T'_\infty)/T'_\infty$ to be at most order one, which means that the buoyancy force ignored in writing (2) is negligible provided that $g r'_0/U'^2_\infty$ is at most order one (on comparison of the buoyancy and viscous forces in the momentum equation), where g is the gravitational acceleration. Alternatively, buoyancy effects are negligible when the Grashof number, G_r, satisfies $G_r \ll R_e$ (as $R_e \to 0$).

The appropriate non-dimensionalized equations governing the flow far from the cylinder which will be referred to as the Oseen or outer region are found from (1)–(4) by defining a new non-dimensional length $s = R_e r$ and using the Oseen non-dimensional form of the pressure, i.e. $P = (p' - p'_\infty)/\rho'_\infty U'^2_\infty$ so that the Oseen equations are

$$\operatorname{div}(\rho\mathbf{q}) = 0, \tag{5}$$

$$\rho(\tfrac{1}{2}\nabla\mathbf{q}^2 - \mathbf{q}\times\operatorname{curl}\mathbf{q}) = -\nabla P + \tfrac{4}{3}\nabla(\mu\operatorname{div}\mathbf{q}) + \nabla(\mathbf{q}.\nabla\mu) - \mathbf{q}\nabla^2\mu + \nabla\mu\times\operatorname{curl}\mathbf{q} - (\operatorname{div}\mathbf{q})\nabla\mu - \operatorname{curl}\operatorname{curl}\mu\mathbf{q}, \tag{6}$$

$$\rho(\mathbf{q}.\nabla)T = P_r^{-1}\operatorname{div}(K\nabla T) + (\gamma' - 1)M^2_\infty(\mu\Phi + \mathbf{q}.\nabla P), \tag{7}$$

and

$$\gamma' M^2_\infty P + 1 = \rho T, \tag{8}$$

where

$$\phi = R_e^2\Phi,$$

and the operators in (5)–(8) are formed with the Oseen variable s.

3. Construction of solution and boundary conditions

We will look for solutions of (1)–(8) of the following form:

Inner	Outer
$\rho = \sum_{n=0}^{\infty} a_n\rho_n + O(R_e);$	$\rho = \sum_{n=0}^{\infty} A_n R_n + O(R_e).$
$p = \sum_{n=0}^{\infty} c_n p_n + O(R_e);$	$P = \sum_{n=0}^{\infty} C_n P_n + O(R_e).$
$T = \sum_{n=0}^{\infty} d_n t_n + O(R_e);$	$T = \sum_{n=0}^{\infty} D_n T_n + O(R_e).$

The velocity $\mathbf{q}$ in the inner region will be defined by

$$\rho\mathbf{q} = \operatorname{curl}(\psi\mathbf{i}_z),$$

where $\mathbf{i}_z$ is the unit vector in the z direction and

$$\psi = \sum_{n=0}^{\infty} b_n\psi_n + O(R_e),$$

so that the continuity equation in this region is always satisfied identically, In the outer region

$$\mathbf{q} = \sum_{n=0}^{\infty} B_n \mathbf{Q}_n + O(R_e).$$

The coefficients of each of the variables, e.g. the coefficients a_n of ρ, are required to possess the property that $a_{n+1}/a_n \to 0$ for all n as R_e, $M_\infty/R_e \to 0$. In addition, the inner and outer solutions for each variable are required to match asymptotically for large r as $R_e \to 0$ or for small s.

The first terms in the expansions for the outer variables are those appropriate to a uniform stream. Hence, on choosing $A_0 = B_0 = C_0 = D_0 = 1$, by definition, $R_0 = 1$, $T_0 = 1$, $P_0 = 0$ and $\mathbf{Q}_0 = \mathbf{i}_x$, where $\mathbf{i}_x$ is the unit vector in the x direction.

If $T'_w/T'_\infty = \Delta$, the boundary conditions on the inner variables at the surface of the cylinder (i.e. $r = 1$) are $T = \Delta$ and $\mathbf{q} = 0$.

4. Solutions for the temperature and density

The temperature field for this problem, valid to first order, is derived in Hodnett (1968) where it is shown that in the Stokes region (near cylinder)

$$T = t_0 - \epsilon^2(\Delta^{m+1} - 1)\,[\ln(\tfrac{1}{4}P_r) + \gamma]\ln r/[(m+1)\,t_0^m], \tag{9}$$

where

$$t_0^{m+1} = \Delta^{m+1} - \epsilon(\Delta^{m+1} - 1)\ln r;$$

the small parameter ϵ is given by $\epsilon = [\ln(1/R_e)]^{-1}$ and γ is Euler's constant. Then $T^{m+1} = \Delta^{m+1} - \delta \ln r$, where

$$\delta = \epsilon(\Delta^{m+1} - 1)\{1 + \epsilon[\ln(\tfrac{1}{4}P_r) + \gamma]\}. \tag{10}$$

In the Oseen region (far from cylinder),

$$T = 1 + \epsilon(\Delta^{m+1} - 1)\,T_1, \tag{11}$$

where

$$T_1 = (m+1)^{-1}\exp(\tfrac{1}{2}P_r s\cos\theta)\,K_0(\tfrac{1}{2}P_r s)$$

and K_0 is the modified Bessel function of the second kind of zero order.

Since we assume $M_\infty/R_e \to 0$ as $R_e \to 0$ and since the expansions for the pressures p and P are in powers of ϵ it is seen from the equations of state (4) and (8) that, neglecting terms of order R_e,

$$\rho T = 1 \tag{12}$$

in both the Stokes and Oseen regions.

5. Solution for the velocity

Since it will be shown that the expansion for the velocity $\mathbf{q}$ is in powers of ϵ. the convection terms in the inner momentum equation (2) can always be neglected, being $O(R_e)$.

As stated in § 3 we let the inner continuity equation (1) be satisfied identically by

$$\rho\mathbf{q} = \operatorname{curl}(\psi\mathbf{i}_z),$$

or, using (12), by

$$\mathbf{q} = T\operatorname{curl}(\psi\mathbf{i}_z), \tag{13}$$

where
$$\psi = \sum_{n=0}^{\infty} b_n \psi_n + O(R_e)$$

and T is given by (9). The zero-order approximation to the inner velocity field is then $\mathbf{q}_0 = b_0 T \operatorname{curl}(\psi_0 \mathbf{i}_z)$.

The outer boundary condition on $\mathbf{q}_0$, that $\mathbf{q}_0 \sim \mathbf{i}_x$ for large r as $R_e \to 0$, suggests ψ_0 should be of the form
$$\psi_0 = \chi_0(r)\sin\theta. \tag{14}$$

After taking the curl of (2) (with convection terms neglected), substitution of expressions (13), (14) and (10) in the resulting equation gives
$$\frac{m\delta}{m+1}\frac{1}{r}\frac{d}{dr}[D\chi_0] - D\left[-\frac{1}{r}\frac{d}{dr}\left(rT^{m+1}\frac{d\chi_0}{dr}\right) + T^{m+1}\frac{\chi_0}{r^2}\right] = 0, \tag{15}$$
where
$$D \equiv \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{1}{r^2}.$$

The boundary conditions in terms of χ_0 are
$$\chi_0 = d\chi_0/dr = 0 \quad \text{at} \quad r = 1 \tag{16}$$
and
$$b_0\chi_0 \sim r \quad \text{for large } r \quad \text{as} \quad R_e \to 0. \tag{17}$$

The solutions of (15) for the limiting cases (i) $m = 0$ (corresponding to constant viscosity and thermal conductivity) and (ii) $(\Delta^{m+1} - 1) = O(\epsilon)$ (corresponding to a slightly heated cylinder) are given in Hodnett (1968). The solution of (15) without any such restrictions is found as follows.

Equation (15) can be rewritten
$$\frac{m\delta}{m+1}\left[r\frac{d}{dr} - 2\right]\left[r\frac{d}{dr}\left(r\frac{d}{dr}\right) - 1\right]\chi_0$$
$$- \left[\left(r\frac{d}{dr} - 2\right)^2 - 1\right]\left[-r\frac{d}{dr}\left(rT^{m+1}\frac{d}{dr}\right) + T^{m+1}\right]\chi_0 = 0. \tag{18}$$

On changing the independent variable from r to $t = T^{m+1}/\delta$ (cf. expression (10)), equation (18) becomes
$$\left(\frac{d}{dt} + 1\right)\left(\frac{m}{m+1}\left[\frac{d}{dt} + 2\right]\left[\frac{d}{dt} - 1\right] + \left[\frac{d}{dt} + 3\right]\left[-\frac{d}{dt}\left(t\frac{d}{dt}\right) + t\right]\right)\chi_0 = 0. \tag{19}$$

Since T^{m+1} is order one and $\delta \to 0$ as $R_e \to 0$, $t \to \infty$ as $R_e \to 0$. We are then interested only in the asymptotic solution for large t of (19). The fourth-order ordinary differential equation (19) possesses four linearly independent solutions of which we need to be able to calculate only three provided they can satisfy the three boundary conditions (16) and (17). By inspection a solution of (19) is
$$\chi_0^a = L_0' e^{-t}, \tag{19a}$$

where L_0' is an arbitrary constant. Then the other three independent solutions of (19) are the three independent solutions of the *third-order* ordinary differential equation
$$\left(\frac{m}{m+1}\left[\frac{d}{dt} + 2\right]\left[\frac{d}{dt} - 1\right] + \left[\frac{d}{dt} + 3\right]\left[-\frac{d}{dt}\left(t\frac{d}{dt}\right) + t\right]\right)\chi_0^b = 0, \tag{20}$$

since, as can easily be verified, χ_0^a is not a solution of (20). Then the full solution, χ_0, of (19) is

$$\chi_0 = \chi_0^a + \chi_0^b.$$

However, we only need two independent solutions of (20) provided they can with solution (19a), satisfy the boundary conditions (16) and (17). We are also only interested in asymptotic solutions of (20) for large t. A method of solution (for large t) is indicated by observing that the second operator in the second term of the equation, i.e.

$$\left[-\frac{d}{dt}\left(t\frac{d}{dt}\right)+t\right]$$

is the operator in the modified Bessel equation of zero order whose asymptotic solution for large t has the form

$$t^{-\frac{1}{2}}[\text{const.}\, e^t + \text{const.}\, e^{-t}].$$

This suggests looking for two asymptotic solutions, valid for large t, of the form

$$\chi_0^b = e^t F(t) + e^{-t} G(t),$$

and seeking a power series solution of the resulting equations for $F(t)$ and $G(t)$. On following this procedure the equations for $F(t)$ and $G(t)$ are

$$\left(\frac{m}{m+1}\frac{d}{dt}\left(\frac{d}{dt}+3\right)+\left[\frac{d}{dt}+4\right]\left[-t\frac{d^2}{dt^2}-(1+2t)\frac{d}{dt}-1\right]\right)F = 0,$$

$$\left(\frac{m}{m+1}\left(\frac{d}{dt}+1\right)\left(\frac{d}{dt}-2\right)+\left(\frac{d}{dt}+2\right)\left[-t\frac{d^2}{dt^2}-(1-2t)\frac{d}{dt}+1\right]\right)G = 0,$$

whose solutions for large t are

$$F = M_0' t^{-\frac{1}{2}}[1 + f_1 t^{-1} + f_2 t^{-2} + O(t^{-3})],$$

where M_0' is an arbitrary constant,

$$f_1 = (5m+2)/16(m+1) \quad \text{and} \quad f_2 = 3(43m^2+46m+12)/[512(m+1)^2],$$

and
$$G = N_0' t^{-\frac{1}{2}(m+1)}[1 + g_1 t^{-1} + g_2 t^{-2} + O(t^{-3})],$$

where N_0' is an arbitrary constant,

$$g_1 = -[8(m+1)^2]^{-1} \quad \text{and} \quad g_2 = (2m+3)(4m^2+6m+3)/[128(m+1)^4].$$

When rewritten in terms of the original independent variable r the solution of (15) then is

$$\chi_0 = L_0 r + M_0 r^{-1} T^{-\frac{1}{2}(m+1)}[1 + f_1 \delta T^{-(m+1)} + f_2 \delta^2 T^{-2(m+1)} + O(\delta^3)]$$
$$+ N_0 r T^{-\frac{1}{2}}[1 + g_1 \delta T^{-(m+1)} + g_2 \delta^2 T^{-2(m+1)} + O(\delta^3)], \quad (21)$$

where $T^{m+1} = \Delta^{m+1} - \delta \ln r$ and L_0, M_0, N_0 are arbitrary constants.

Applying the boundary conditions (16) to (21) then gives

$$M_0 = -\frac{L_0 \delta}{4(m+1)\Delta^{\frac{1}{2}(m+1)}}\left[1 - \frac{(m+6)\delta}{16(m+1)\Delta^{m+1}} + O(\delta^2)\right], \quad (22)$$

$$N_0 = -L_0 \Delta^{\frac{1}{2}}\left[1 - \frac{(2m+1)\delta}{8(m+1)^2\Delta^{m+1}} + O(\delta^2)\right]. \quad (23)$$

By equation (21) on using (23),

$$b_0\chi_0 \sim b_0 r(L_0 - L_0\Delta^{\frac{1}{2}}),$$

for large r as $R_e \to 0$, so that boundary condition (17) is satisfied on choosing $b_0 = 1$, if

$$L_0 = 1/(1-\Delta^{\frac{1}{2}}). \tag{24}$$

It is worth noting here that ψ_0 given by expressions (14) and (21)–(24) is not singular at the outer edge of the Stokes region and is therefore, a uniformly valid zero-order solution for the velocity field satisfying both the no-slip condition at the cylinder and the uniform stream condition at infinity. This contrasts strongly with the corresponding incompressible problem of slow flow past an unheated circular cylinder when ψ_0 needs to be multiplied by ϵ to ensure proper behaviour of the velocity field at the outer edge of the Stokes region. This result was also noted in Hodnett (1968) for the case $m = 0$.

The pressure p_0 can now be calculated by integrating the momentum equation (2).

As shown in Hodnett (1968) the expansion in the outer region for the temperature T is given by (11), and the expansion for the density ρ in this region is

$$\rho = 1+\epsilon(\Delta^{m+1}-1)R_1, \tag{25}$$

where $R_1 = -T_1$.

The form of the first-order outer continuity and momentum equations can then be found when the outer expansion for the velocity $\mathbf{q}$ is known. The appropriate form of the outer expansion for $\mathbf{q}$ becomes apparent when $\mathbf{q}_0$ is expressed in terms of the Oseen variable s. Equation (13) gives

$$\mathbf{q}_0 = T\left[\frac{1}{r}\frac{\partial\psi_0}{\partial\theta}\mathbf{i}_r - \frac{\partial\psi_0}{\partial r}\mathbf{i}_\theta\right]. \tag{26}$$

For large r as $R_e \to 0$, it is seen from (9) that

$$T \sim 1-\epsilon(\Delta^{m+1}-1)(m+1)^{-1}[\ln s + \ln(\tfrac{1}{4}P_r)+\gamma]+O(\epsilon^2), \tag{27}$$

while from (14) and (21)–(24),

$$\frac{1}{r}\frac{\partial\psi_0}{\partial\theta} \sim \cos\theta\left(1-\frac{\epsilon\Delta^{\frac{1}{2}}(\Delta^{m+1}-1)}{2(1-\Delta^{\frac{1}{2}})(m+1)}\left[\ln s+\ln(\tfrac{1}{4}P_r)+\gamma-\frac{2m+1+\Delta^{m+1}}{4(m+1)\Delta^{m+1}}\right]+O(\epsilon^2)\right), \tag{28}$$

$$\frac{\partial\psi_0}{\partial r} \sim \sin\theta\left(1-\frac{\epsilon\Delta^{\frac{1}{2}}(\Delta^{m+1}-1)}{2(1-\Delta^{\frac{1}{2}})(m+1)}\left[\ln s+\ln(\tfrac{1}{4}P_r)+\gamma+\frac{(4m+3)\Delta^{m+1}-(2m+1)}{(4m+1)\Delta^{m+1}}\right]+O(\epsilon^2)\right). \tag{29}$$

Equations (26)–(29) together indicate that the outer expansion for the velocity $\mathbf{q}$ should be of the form

$$\mathbf{q} = \mathbf{i}_x+\epsilon\mathbf{Q}_1 \quad (\text{i.e. } B_1 = \epsilon). \tag{30}$$

It is then apparent that the outer expansion for the pressure P should be

$$P = \epsilon P_1 \quad (\text{i.e. } C_1 = \epsilon). \tag{31}$$

The first-order outer continuity equation by (30), (25) and (5) then is

$$\operatorname{div}\mathbf{Q}_1-(\Delta^{m+1}-1)\frac{\partial T_1}{\partial x}=0, \tag{32}$$

while the first-order outer momentum equation by (31), (30), (25), (11) and (6) is

$$\frac{\partial \mathbf{Q}_1}{\partial x}=-\nabla(P_1-\tfrac{4}{3}\operatorname{div}\mathbf{Q}_1)-\operatorname{curl}\operatorname{curl}\mathbf{Q}_1. \tag{33}$$

Since the energy equation (7) gives $\partial T_1/\partial x = P_r^{-1}\nabla^2 T_1$, then (32) becomes

$$\operatorname{div}[\mathbf{Q}_1-P_r^{-1}(\Delta^{m+1}-1)\nabla T_1]=0,$$

which is satisfied by putting

$$\mathbf{Q}_1=P_r^{-1}(\Delta^{m+1}-1)\nabla T_1+\operatorname{curl}(\Psi_1\mathbf{i}_z). \tag{34}$$

After taking the curl of (33), substitution of expression (34) in the resulting equation gives

$$\left[\frac{\partial}{\partial x}+\operatorname{curl}\operatorname{curl}\right][\operatorname{curl}\operatorname{curl}(\Psi_1\mathbf{i}_z)]=0,$$

where

$$\operatorname{curl}\operatorname{curl}(\Psi_1\mathbf{i}_z)=\left(-\left[\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2}{\partial\theta^2}\right]\Psi_1\right)\mathbf{i}_z$$
$$=X_1(s,\theta)\,\mathbf{i}_z. \tag{35}$$

The equation

$$\left(\frac{\partial}{\partial x}+\operatorname{curl}\operatorname{curl}\right)X_1=0$$

has the solution

$$X_1=\exp(\tfrac{1}{2}s\cos\theta)\sum_{n=1}^{\infty}E_nK_n(\tfrac{1}{2}s)\sin n\theta,$$

which is bounded at infinity. The E_n are arbitrary constants and the K_n are modified Bessel functions of the second kind. The constants E_n can be determined by the condition that

$R_e^{-1}\operatorname{curl}\mathbf{q}_0$ must asymptotically $\to \epsilon\operatorname{curl}\mathbf{Q}_1$ for large r as $R_e\to 0$.

By using (26), (9), (14) and (21), for large r as $R_e\to 0$,

$$R_e^{-1}\operatorname{curl}\mathbf{q}_0\sim\epsilon s^{-1}\frac{\Delta^{m+1}-1}{(m+1)(1-\Delta^{\frac{1}{2}})}\sin\theta\,\mathbf{i}_z. \tag{36}$$

By (34) and (35)

$$\epsilon\operatorname{curl}\mathbf{Q}_1=\epsilon\exp(\tfrac{1}{2}s\cos\theta)\sum_{n=1}^{\infty}E_nK_n(\tfrac{1}{2}s)\sin n\theta\mathbf{i}_z. \tag{37}$$

For expression (37) to tend asymptotically to expression (36) for small s.

$$E_n=0\quad\text{for}\quad n\geqslant 2.$$

while

$$E_1=\frac{\Delta^{m+1}-1}{2(m+1)(1-\Delta^{\frac{1}{2}})}. \tag{38}$$

Equation (35) then becomes

$$\left[-\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right)-\frac{1}{s^2}\frac{\partial^2}{\partial\theta^2}\right]\Psi_1=E_1\exp(\tfrac{1}{2}s\cos\theta)K_1(\tfrac{1}{2}s)\sin\theta, \tag{39}$$

where E_1 is given by (38). Let

$$\Psi_1 = \Psi_{1c} + \Psi_{1p},$$

where Ψ_{1c} is the complementary function and Ψ_{1p} is the particular integral of (39).

The particular integral of (39) is

$$\Psi_{1p} = E_1 \sum_{n=1}^{\infty} \phi_n(\tfrac{1}{2}s)\, s \frac{\sin n\theta}{n}, \tag{40}$$

where $\phi_n = 2K_1 I_n + K_0(I_{n+1} + I_{n-1})$, and I_n is the modified Bessel function of the first kind of order n. The second term, $\mathbf{Q}_1$, in the outer expansion for the velocity $\mathbf{q}$ is, on using (34), (40) and (11),

$$\left[\frac{1}{s}\frac{\partial \Psi_{1c}}{\partial \theta} + E_1 \sum_{n=1}^{\infty} \phi_n(\tfrac{1}{2}s)\cos n\theta + \frac{\Delta^{m+1}-1}{2(m+1)} \exp(\tfrac{1}{2}P_r s\cos\theta)\,[\cos\theta K_0(\tfrac{1}{2}P_r s) - K_1(\tfrac{1}{2}P_r s)]\right]\mathbf{i}_r + \left[-\frac{\partial \Psi_{1c}}{\partial s} - E_1 \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}[\phi_n(\tfrac{1}{2}s) + \tfrac{1}{2}s\phi_n'(\tfrac{1}{2}s)] - \frac{\Delta^{m+1}-1}{2(m+1)} \sin\theta \exp(\tfrac{1}{2}P_r s\cos\theta)\, K_0(\tfrac{1}{2}P_r s)\right]\mathbf{i}_\theta, \tag{41}$$

where $\phi_n'(s) = d/ds(\phi_n)$.

For small s

$$\phi_1(\tfrac{1}{2}s) > \phi_2 > \phi_3 \ldots,$$

while

$$\phi_1(\tfrac{1}{2}s) + \tfrac{1}{2}s\phi_1'(\tfrac{1}{2}s) > \phi_2 + \tfrac{1}{2}s\phi_2' > \phi_3 + \tfrac{1}{2}s\phi_3' \ldots$$

and

$$\phi_1(\tfrac{1}{2}s) \sim -\ln s + [\ln 4 - \gamma + 1],$$

while

$$\phi_1(\tfrac{1}{2}s) + \tfrac{1}{2}s\phi_1'(\tfrac{1}{2}s) \sim -\ln s + [\ln 4 - \gamma]. \tag{42}$$

An inspection of (26)–(29) which give $\mathbf{q}_0$ for large r as $R_e \to 0$ and use of (41) and (42) which give $\mathbf{Q}_1$ for small s, show that for the first two terms in the outer expansion for the velocity

$$\mathbf{i}_x + \epsilon \mathbf{Q}_1$$

to match $\mathbf{q}_0$ for small s,

$$\frac{\partial \Psi_{1c}}{\partial \theta} \to \frac{\Delta^{m+1}-1}{m+1} P_r^{-1} \quad \text{(for small } s\text{)},$$

and

$$\frac{\partial \Psi_{1c}}{\partial s} \to 0 \quad \text{(for small } s\text{)}. \tag{43}$$

The condition that $\mathbf{Q}_1 \to 0$ (as $s \to \infty$) requires

$$\frac{1}{s}\frac{\partial \Psi_{1c}}{\partial \theta},\quad \frac{\partial \Psi_{1c}}{\partial s} \to 0 \quad \text{as} \quad s \to \infty. \tag{44}$$

The complementary function, Ψ_{1c}, of (39) satisfying boundary conditions (43) and (44) then is

$$\Psi_{1c} = (\Delta^{m+1} - 1) P_r^{-1}(m+1)^{-1}\theta. \tag{45}$$

The form of the next term in the inner expansion for $\mathbf{q}$ is found by expanding for small s the first two terms, $\mathbf{i}_x + \epsilon \mathbf{Q}_1$, in the outer expansion for the velocity. For small s, using (41), (45) and (42),

$$\mathbf{i}_x + \epsilon \mathbf{Q}_1 \sim \left(\mathbf{1} + \frac{\epsilon(\Delta^{m+1}-1)}{2(m+1)(1-\Delta^{\frac{1}{2}})}[-\mathbf{ln}\, \boldsymbol{s} + \ln 4 - \gamma + 1] - \frac{\epsilon(\Delta^{m+1}-1)}{2(m+1)} \right.$$
$$\left. \times [\mathbf{ln}\, \boldsymbol{s} + \ln P_r - \ln 4 + \gamma + 1] \right) \cos\theta \mathbf{i}_r + \left(-\mathbf{1} - \frac{\epsilon(\Delta^{m+1}-1)}{2(m+1)(1-\Delta^{\frac{1}{2}})} \right.$$
$$\left. \times [-\mathbf{ln}\, \boldsymbol{s} + \ln 4 - \gamma] + \frac{\epsilon(\Delta^{m+1}-1)}{2(m+1)}[\mathbf{ln}\, \boldsymbol{s} + \ln P_r - \ln 4 + \gamma] \right) \sin\theta \mathbf{i}_\theta, \qquad (46)$$

where an inspection of expressions (26)–(29) which give $\mathbf{q}_0$ for large r as $R_e \to 0$ shows that the bold face terms in expression (46) already match terms in the expansion for $\mathbf{q}_0$ (for large r as $R_e \to 0$). For the inner expansion for $\mathbf{q}$ to satisfy the asymptotic matching condition for large r as $R_e \to 0$ then requires, using expression (46), that

$$\psi = \psi_0 + \epsilon \psi_1 \quad \text{(i.e. } b_1 = \epsilon\text{)},$$

where

$$\psi_1 = \chi_1(r) \sin\theta,$$

and as before

$$\rho \mathbf{q} = \text{curl}\,(\psi \mathbf{i}_z). \qquad (47)$$

After taking the curl of the momentum equation (2) (with the convection terms neglected), substitution of expressions (47), (12) and (10) in the resulting equation gives χ_1 satisfying the same differential equation as χ_0 does, i.e. (15). The solution, χ_1, of this equation is given by expression (21) with χ_1 replacing χ_0 and new arbitrary constants L_1, M_1, N_1 replacing L_0, M_0, N_0. As previously, the boundary conditions at the cylinder

$$\chi_1 = d\chi_1/dr = 0 \quad \text{at} \quad r = 1,$$

determine M_1 and N_1 in terms of L_1. These relations are given by (22) and (23) with L_1, M_1, N_1 replacing L_0, M_0, N_0.

An inspection of expressions (46), (47) and (26)–(29) shows that the outer boundary on the inner expansion for $\mathbf{q}$ gives, after some reduction,

$$\frac{1}{r}\chi_1 \sim \frac{\Delta^{m-1}-1}{2(m+1)(1-\Delta^{\frac{1}{2}})}\left[\ln P_r + \frac{\Delta^{\frac{1}{2}}[(4m+3)\Delta^{m+1}-(2m+1)]}{4(m+1)\Delta^{m+1}}\right], \qquad (48)$$

for large r as $R_e \to 0$. Expressions (21)–(23) for χ_1 give,

$$\chi_1 \sim rL_1(1-\Delta^{\frac{1}{2}}),$$

for large r as $R_e \to 0$, so that expression (48) is satisfied when

$$L_1 = \frac{\Delta^{m+1}-1}{2(m+1)(\Delta^{\frac{1}{2}}-1)^2}\left[\ln P_r + \frac{\Delta^{\frac{1}{2}}[(4m+3)\Delta^{m+1}-(2m+1)]}{4(m+1)\Delta^{m+1}}\right]. \qquad (49)$$

The stream function ψ for the inner velocity field then is

$$\psi = \sin\theta(\chi_0 + \epsilon\chi_1),$$

where χ_i $(i = 0, 1)$ are given by (21)–(23) with L_i, M_i, N_i $(i = 0, 1)$ replacing L_0, M_0, N_0 and L_0, L_1 are given by (24) and (49) respectively.

6. The drag on the cylinder

When the drag on the cylinder is denoted by D, the drag coefficient C_D, defined by

$$C_D = D/[\tfrac{1}{2}\rho'_\infty(U'_\infty)^2\, 2r'_0],$$

is

$$C_D = \frac{1}{R_e}\int_0^{2\pi}\left[\left(-p+2\mu\frac{\partial q_r}{\partial r}\right)\cos\theta - \mu\sin\theta\left(r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right)+\frac{1}{r}\frac{\partial q_r}{\partial\theta}\right)\right]d\theta,$$

where the integral is evaluated at $r = 1$.

The pressure p is calculated by integrating the momentum equation (2) now that the temperature T and the stream function ψ for the inner velocity field are known. It is found that

$$C_D = \frac{2\pi\epsilon(\Delta^{m+1}-1)}{(m+1)(\Delta^{\frac{1}{2}}-1)R_e}\left(1+\epsilon\left[\ln(\tfrac{1}{4}P_r)+\gamma-\frac{(\Delta^{m+1}-1)\ln P_r}{2(m+1)(\Delta^{\frac{1}{2}}-1)}\right.\right.$$
$$\left.\left.-\frac{(\Delta^{m+1}-1)\Delta^{\frac{1}{2}}[(4m+3)\Delta^{m+1}-(2m+1)]}{8(m+1)^2(\Delta^{\frac{1}{2}}-1)\Delta^{m+1}}\right]+O(\epsilon^2)\right). \quad (50)$$

Expression (50) for C_D reduces to the equivalent expressions for C_D given by Hodnett (1968) for the limiting cases (i) $m = 0$ and (ii) $(\Delta^{m+1}-1) = O(\epsilon)$ and to the incompressible result for the drag coefficient on an unheated cylinder given in Lamb (1945, p. 616) on putting $m = 0$ and letting $\Delta \to 1$ in expression (50).

7. Discussion of results

The zero-order solution, ψ_0, for the Stokes region stream function is not singular at the outer edge of the Stokes region in distinct contrast with the corresponding ψ_0 for the unheated cylinder problem which needs to be multiplied by ϵ to ensure proper behaviour of the velocity field at the outer edge of the Stokes region. This interesting property of the Stokes region stream function for the heated cylinder was also exhibited by the stream function, ψ_0, given in Hodnett (1968) which was however calculated on the assumption that the viscosity and thermal conductivity of the gas are constant but the density is allowed its proper variation. That the Stokes region stream function satisfies the outside boundary condition is then directly attributable to the density variation of the gas. Physically this can be explained by the fact that heating the cylinder decreases the gas density near the cylinder and in order to conserve mass the speed of the gas at a point at given distance from the heated cylinder is greater than the speed of the gas at the same point when flowing around an unheated cylinder.

The effect of heating the cylinder is to increase the drag significantly. This can be seen by comparing the drag coefficient, C_D, for a heated and unheated cylinder. For the unheated cylinder, to zero order, the drag coefficient $C_D|_{\text{inc}} = 4\pi\epsilon/R_e$, (as $R_e \to 0$), from Lamb (1945). For the heated cylinder, to zero order, the drag coefficient

$$C_D|_{\text{comp}} = \frac{2\pi\epsilon}{R_e}\frac{\Delta^{m+1}-1}{(m+1)(\Delta^{\frac{1}{2}}-1)}, \quad (\text{as } R_e \to 0), \quad \text{from (50)}.$$

For the case $m = 1$ (i.e. viscosity, thermal conductivity αT),

$$\frac{C_D|_{\text{comp}}}{C_D|_{\text{inc}}} = \frac{(\Delta+1)(\Delta^{\frac{1}{2}}+1)}{4} \approx 1{\cdot}81 \quad (\text{when } \Delta = 2).$$

Thus, when the temperature at the cylinder is twice the temperature at infinity, the drag on the heated cylinder is almost twice the drag on a similar unheated cylinder.

The conditions under which the solution for the velocity field (when the heating factor is order one) given in Hodnett (1968) is not adequate are clearly seen by comparing C_D evaluated when $m = 0$ with $C_D|_{\text{comp}}$. When $m = 0$, from (50), the drag coefficient (to zero order) is

$$C_D|_{m=0} = \frac{2\pi\epsilon}{R_e}(\Delta^{\frac{1}{2}}+1) \quad (\text{as } R_e \to 0).$$

Then $$\frac{C_D|_{\text{comp}}}{C_D|_{m=0}} = \frac{\Delta^{m+1}-1}{(m+1)(\Delta-1)} = \frac{3}{2} \quad \text{for} \quad m = 1 \quad \text{and} \quad \Delta = 2.$$

Thus if the physical conditions of the problem are such that $m = 1$ describes the variation of the gas properties it is seen that the velocity field given in Hodnett (1968) is then quite inadequate and the velocity field given here must be used. Of course, if the gas property variation can be adequately described by $m \approx 0$ then the velocity field (when the heating factor is order one) given in Hodnett (1968) is sufficient.

REFERENCES

Chang, I.-D. 1965 *Z. angew. Math. Phys.* **16**, 449.

Cole, J. D. & Roshko, A. 1954 In *Proceedings of Heat Transfer and Fluid Mechanics Institute*, pp. 13–21. Stanford University Press.

Dennis, S. C. R., Hudson, J. D. & Smith, N. 1968 *Phys. Fluids*, **11**, 933.

Hieber, C. A. & Gebhart, B. 1968 *J. Fluid Mech.* **32**, 21.

Hodnett, P. F. 1968 *Phys. Fluids*, **11**, 1636.

Kaplun, S. 1957 *J. Math. Mech.* **6**, 595.

Kassoy, D. R. 1967 *Phys. Fluids*, **10**, 938.

Kassoy, D. R., Adamson, T. C. & Messiter, A. F. 1966 *Phys. Fluids*, **9**, 671.

Lamb, H. 1945 *Hydrodynamics*, 6th ed. New York: Dover.

Proudman, I. & Pearson, J. R. A. 1957 *J. Fluid. Mech.* **2**, 237

Rimmer, P. L. 1968 *J. Fluid Mech.* **32**, 1.

Wood, W. W. 1968 *J. Fluid Mech.* **32**, 9.